

THE BEHAVIOR OF HECKE'S L-FUNCTION OF REAL QUADRATIC FIELDS AT $s = 0$

BYUNGHEUP JUN AND JUNGYUN LEE

ABSTRACT. For a family of real quadratic fields $\{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n \in \mathbb{N}}$, a Dirichlet character χ modulo q and prescribed ideals $\{\mathfrak{b}_n \subset K_n\}$, we investigate the linear behaviour of the special value of partial Hecke's L-function $L_{K_n}(s, \chi_n := \chi \circ N_{K_n}, \mathfrak{b}_n)$ at $s = 0$. We show that for $n = qk + r$, $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ can be written as

$$\frac{1}{12q^2}(A_\chi(r) + kB_\chi(r)),$$

where $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi(1), \chi(2), \dots, \chi(q)]$ if a certain condition on \mathfrak{b}_n in terms of its continued fraction is satisfied. Furthermore, we write precisely $A_\chi(r)$ and $B_\chi(r)$ using values of the Bernoulli polynomials. We describe how the linearity is used in solving class number one problem for some families and recover the proofs in some cases. Finally, we list some families of real quadratic fields with the linearity.

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1. INTRODUCTION

In this paper, we are mainly concerned with linear behaviour of the special values of Hecke's L -function at $s = 0$ for families of real quadratic fields.

Let $\{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n \in \mathbb{N}}$ be a family of real quadratic fields where $f(n)$ is a positive square free integer for each n . For example $f(x)$ can be a polynomial with integer coefficients.

For a Dirichlet character χ modulo q , we have a ray class character $\chi_n := \chi \circ N_{K_n}$ for each n . Fixing an ideal \mathfrak{b}_n in K_n for each n , one obtains an indexed family of partial Hecke L -functions $\{L_{K_n}(s, \chi_n, \mathfrak{b}_n)\}$, where the partial Hecke's L -function for (K, χ, \mathfrak{b}) is defined as

$$L_K(s, \chi, \mathfrak{b}) := \sum_{\substack{\mathfrak{a} \sim \mathfrak{b} \\ \text{integral} \\ (q, \mathfrak{a})=1}} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s}.$$

and $\mathfrak{a} \sim \mathfrak{b}$ means that $\mathfrak{a} = \alpha \mathfrak{b}$ for totally positive $\alpha \in K$.

Roughly speaking, if $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ can be written as linear polynomial in k with coefficients depending only on r for $n = qk + r$, we say that $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is linear.

Definition 1.1 (Linearity). *When the special values of $L_{K_n}(s, \chi_n, \mathfrak{b}_n)$ at $s = 0$ is expressed as*

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{12q^2} (A_\chi(r) + kB_\chi(r))$$

for $n = qk + r$, $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi(1), \chi(2), \dots, \chi(q)]$, we say that $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is **linear**.

The “linearity” is originally observed by Biró in his proof of Yokoi's conjecture([2]).

Theorem 1.2 (Yokoi's conjecture solved by Biró). *If the class number of $\mathbb{Q}(\sqrt{n^2 + 4})$ is 1 then $n \leq 17$.*

In Yokoi's conjecture, we take $K_n = \mathbb{Q}(\sqrt{n^2 + 4})$ and $\mathfrak{b}_n = O_{K_n}$. In page 88, 89 of [2], Biró expressed the special value of Hecke's L -function for (K_n, χ_n, O_{K_n}) at $s = 0$ for $n = qk + r$

$$(1) \quad L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{q} (A_\chi(r) + kB_\chi(r)),$$

where

$$A_\chi(r) = \sum_{0 \leq C, D \leq q-1} \chi(D^2 - C^2 - rCD) \left\lceil \frac{rC - D}{q} \right\rceil (C - q),$$

$$B_\chi(r) = \sum_{0 \leq C, D \leq q-1} \chi(D^2 - C^2 - rCD) C(C - q).$$

When K_n is of class number 1, the unique ideal class can be represented by any ideal \mathfrak{b}_n . *A priori* the partial Hecke L -function equals the total Hecke L -function up to multiplication by 2 (ie.

$$L_{K_n}(0, \chi_n) = c L_{K_n}(0, \chi_n, O_{K_n})$$

for c the number of narrow ideal classes).

From this identification, one can find the residue of n by sufficiently many primes p for which the class number of $\mathbb{Q}(\sqrt{n^2 + 4})$ is one. Moreover, from the linearity, this residue depends only on r . Consequently, one can tell whether p inerts or not in $\mathbb{Q}(\sqrt{n^2 + 4})$. As we have a bound for a smaller prime to inert depending on n , finally we have enough conditions to list all K_n of class number 1.

Later in diverse works of Biró, Byeon, Kim and the second named author ([3],[7],[8],[12],[11]), other families $(K_n, \chi_n, \mathfrak{b}_n)$ that has linearity have been discovered. Similarly, developing Biro's method, one can solve the associated class number one problems.

In this paper, we give a criterion for $(K_n, \chi_n, \mathfrak{b}_n)$ to have the linearity of the values $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ in terms of the continued fraction expression of $\delta(n)$ where $\mathfrak{b}_n^{-1} = [1, \delta(n)] := \mathbb{Z} + \delta(n)\mathbb{Z}$. Let $[[a_0, a_1, \dots, a_n]]$ be the purely periodic minus continued fraction

$$[a_0, a_1, a_2, \dots, a_n, a_0, a_1, \dots],$$

where

$$[a_0, a_1, a_2, \dots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}.$$

Our main theorem is as follows:

Theorem 1.3 (Linearity Criterion). *Let $\{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n \in \mathbb{N}}$ be a family of real quadratic fields where $f(n)$ is a positive square free integer for each n . Let χ be a Dirichlet character modulo q for a positive integer q and χ_n be a ray class character modulo q defined by $\chi \circ N_{K_n}$. Suppose \mathfrak{b}_n is an integral ideal relatively prime to q such that $\mathfrak{b}_n^{-1} = [1, \delta(n)]$. Assume the continued fraction expansion of $\delta(n) - 1$*

$$\delta(n) - 1 = [[a_0(n), a_1(n), \dots, a_{s-1}(n)]]$$

is purely periodic and of a fixed length s independent of n and $a_i(n) = \alpha_i n + \beta_i$ for some fixed $\alpha_i, \beta_i \in \mathbb{Z}$.

If $N_{K_n}(\mathbf{b}_n(C + D\delta(n)))$ modulo q is a function only depending on C , D and r for $n = qk + r$, then $L_{K_n}(0, \chi_n, \mathbf{b}_n)$ is linear.

Furthermore, we give a precise description of $A_\chi(r)$ and $B_\chi(r)$ using values of the Bernoulli polynomials (Proposition 3.8). From this description, for n with $h(K_n) = 1$, as in Biró's case, one can compute the residue of n modulo p depending on the mod- q residue r of n . There are possibly many (q, p) pairs. The more pairs of (q, p) we have, the more we can restrict possible n . There are known many families of which class number one problem can be solved in this way. Many of known results can be recovered by ensuring the linearity from continued fraction expansion and finding enough (q, p) .

There are still other families of real quadratic fields with linearity whose class number one problems are not yet answered. Morally, once we obtain reasonable class number one criterion, finding sufficiently many (q, p) -pairs should solve it.

This paper is composed as follows. In Section 2, we describe the special value at $s = 0$ of the partial Hecke L-function in terms of values of the Bernoulli polynomials. Section 3 is devoted to the proof of our main theorem. In Section 4, Biró's method is sketched as a prototype to apply the linearity. Finally in Section 5, we finish this paper with a possible generalization of the linearity criterion to polynomial of higher order.

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NOTATIONS AND CONVENTIONS

Throughout this article, we keep the following general notations and conventions. If we find it necessary, we rewrite the notations in concrete terms at the place where it is used.

- (1) K is a real quadratic field.
- (2) For a real quadratic field K , we fix an embedding $\iota : K \rightarrow \mathbb{R}$. If there is no danger of confusion, we denote $\iota(\alpha)$ by an element $\alpha \in K$. α' denotes the conjugate of α as well as $\iota(\alpha')$.
- (3) For $\alpha \in K$, $N_K(\alpha)$ denotes the norm of α over \mathbb{Q} . If there is no danger of confusion, we simply write $N(\alpha)$ to denote $N_K(\alpha)$.

For an integral ideal \mathfrak{a} of K , $N(\mathfrak{a})$ denotes the norm of \mathfrak{a} defined to be $[\mathfrak{o}_K, \mathfrak{a}]$.

- (4) For two linearly independent elements $\alpha, \beta \in K$ as a vector space over \mathbb{Q} , $[\alpha, \beta]$ denotes the lattice (ie. free abelian group) generated by α and β . A fractional ideal \mathfrak{a} of K seen as a lattice is denoted by $[\alpha, \beta]$ if $\{\alpha, \beta\}$ is a free basis of \mathfrak{a} .
- (5) For a subset A of K , we denote A^+ the set of totally positive elements in A .
- (6) χ is a fixed Dirichlet character of modulus q .
- (7) For a real number x ,

$$\langle x \rangle := \begin{cases} x - [x], & \text{for } x \notin \mathbb{Z} \\ 1, & \text{for } x \in \mathbb{Z} \end{cases}$$

Equivalently, $\langle - \rangle$ is the unique composition $\mathbb{R} \xrightarrow{\text{mod } \mathbb{Z}} \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ that is identity on $(0, 1]$.

- (8) For a real x , $[x]_1 := x - \langle x \rangle$.
- (9) For an integer m , $\langle m \rangle_q$ denotes the residue of m in $[1, q]$ by q (ie. $m = qk + \langle m \rangle_q$ for $k \in \mathbb{Z}$, $\langle m \rangle_q \in [1, q] \cap \mathbb{Z}$).
- (10) $[a_0, a_1, a_2, \dots]$ for positive integers a_i denotes the usual continued fraction:

$$[a_0, a_1, a_2, \dots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

$[a_0, a_1, \dots, a_{i-1}, \overline{a_i, a_{i+1}, \dots, a_{i+j}}]$ denotes the continued fraction with periodic part $(a_i, a_{i+1}, \dots, a_{i+j})$.

$[[a_0, a_1, \dots, a_n]]$ is the purely periodic continued fraction

$$[a_0, a_1, \dots, a_n, a_0, a_1, \dots].$$

- (11) (a_0, a_1, a_2, \dots) denotes the minus continued fraction:

$$(a_0, a_1, a_2, \dots) := a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \dots}}$$

$((a_0, a_1, \dots, a_n))$ is the purely periodic minus continued fraction:

$$(a_0, a_1, a_2, \dots, a_n, a_0, a_1, \dots)$$

- (12) For an integer s , $\mu(s) = 1$ (resp. $\frac{1}{2}$) if s is odd (resp. even).

2. PARTIAL HECKE L -FUNCTION

Throughout this section, K denotes a real quadratic field and \mathfrak{b} is a fixed integral ideal of K relatively prime to q .

A *ray class character* modulo q is a homomorphism

$$\chi : I_K(q)/P_K(q) \rightarrow \mathbb{C}^*,$$

where $I_K(q)$ is a group of fractional ideals of K which is relatively prime to q and $P_K(q)$ is a subgroup of principal ideals (α) for totally positive $\alpha \equiv 1 \pmod{q}$.

Throughout this section, \mathfrak{b} is an integral ideal such that $\mathfrak{b}^{-1} = [1, \delta]$ for $\delta \in K$ satisfying $0 < \delta' < 1$ and $\delta > 2$.

Define

$$F := \{(C, D) \in \mathbb{Z}^2 | 0 \leq C, D \leq q-1, ((C + D\delta)\mathfrak{b}, q) = 1\}.$$

Let E^+ (resp. E_q^+) be the set of totally positive units (resp. the set of totally positive units congruent to 1 mod q) in K . Then E^+ acts on the set F by the rule

$$\epsilon * (C + D\delta) = C' + D'\delta$$

where $\epsilon \cdot (C + D\delta) + q\mathfrak{b}^{-1} = C' + D'\delta + q\mathfrak{b}^{-1}$ for $\epsilon \in E^+$.

Lemma 2.1. (C, D) in F is fixed by the action of ϵ if and only if ϵ is in E_q^+ .

Proof. (C, D) is fixed by $\epsilon \in E^+$ if and only if $(C + D\delta)(\epsilon - 1) \in q\mathfrak{b}^{-1}$. Since $(\mathfrak{b}(C + D\delta), q) = 1$, the condition $(C + D\delta)(\epsilon - 1) \in q\mathfrak{b}^{-1}$ is equivalent to $\epsilon \equiv 1 \pmod{q}$. \square

Lemma 2.2. Suppose $0 \leq C, D \leq q-1$. Then the following are equivalent:

- (1) (C, D) is in F .
- (2) For every $\alpha \in \frac{C+D\delta}{q} + \mathfrak{b}^{-1}$, the ideal $q\alpha\mathfrak{b}$ is relatively prime to q .
- (3) For a $\alpha \in \frac{C+D\delta}{q} + \mathfrak{b}^{-1}$, the ideal $q\alpha\mathfrak{b}$ is relatively prime to q .

Proof. Suppose that $(q, (C + D\delta)\mathfrak{b}) = 1$.

We have $\frac{q\alpha}{C+D\delta} \in 1 + \frac{q}{C+D\delta}\mathfrak{b}^{-1}$ for $\alpha \in \frac{C+D\delta}{q} + \mathfrak{b}^{-1}$. Thus $(q, \mathfrak{b}(C + D\delta)) = 1$ implies that

$$\frac{q\alpha}{C + D\delta} \equiv 1 \pmod{q}.$$

Since

$$q\mathfrak{b}\alpha = \mathfrak{b}(C + D\delta)\frac{q\alpha}{C + D\delta},$$

we have

$$(q\mathfrak{b}\alpha, q) = 1.$$

If $(q, (C + D\delta)\mathfrak{b}) \neq 1$, then $(q, q\mathfrak{b}\alpha) \neq 1$ for $\alpha \in \frac{C+D\delta}{q} + \mathfrak{b}^{-1}$, since for $\alpha \in \frac{C+D\delta}{q} + \mathfrak{b}^{-1}$, we have

$$q\mathfrak{b}\alpha \subset (C + D\delta)\mathfrak{b} + qO_K.$$

□

Let $F' = F/E^+$ be the orbit space by the action of E^+ on F . Let \tilde{F}' a fundamental set of F' . Let ϵ be the totally positive fundamental unit. The order of the action of ϵ is $\lambda := [E^+ : E_q^+]$ by Lemma 2.1. Then we can decompose F as follows:

$$(2) \quad F = \bigsqcup_{i=0}^{\lambda-1} \epsilon^i \tilde{F}'.$$

According to this decomposition of F , we can decompose further the partial Hecke's L -function:

Proposition 2.3. *Let q be a positive integer. For an ideal $\mathfrak{b} \subset K$ relatively prime to q and a ray class character χ modulo q , we have*

$$\begin{aligned} L_K(s, \chi, \mathfrak{b}) &= \sum_{\substack{\mathfrak{a} \sim \mathfrak{b} \\ \text{integral} \\ (q, \mathfrak{a})=1}} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s} \\ &= \sum_{(C, D) \in \tilde{F}'} \chi((C + D\delta)\mathfrak{b}) \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+} N(q\mathfrak{b}\alpha)^{-s}. \end{aligned}$$

Proof. For $\alpha_1, \alpha_2 \in (q^{-1}\mathfrak{b}^{-1})^+$, $q\alpha_1\mathfrak{b} = q\alpha_2\mathfrak{b}$ if and only if $\alpha_1/\alpha_2 \in E^+$.

So we have

$$\sum_{\substack{\mathfrak{a} \sim \mathfrak{b} \\ \text{integral} \\ (q, \mathfrak{a})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{\substack{\mathfrak{a} \sim q\mathfrak{b} \\ \text{integral} \\ (q, \mathfrak{a})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{\substack{\alpha \in (q^{-1}\mathfrak{b}^{-1})^+ / E^+ \\ (q, q\alpha\mathfrak{b})=1}} \frac{\chi(q\alpha\mathfrak{b})}{N(q\alpha\mathfrak{b})^s}$$

We also have for a totally positive fundamental unit $\epsilon > 1$

$$\begin{aligned} \sum_{\substack{\alpha \in (q^{-1}\mathfrak{b}^{-1})^+ / E_q^+ \\ (q, q\mathfrak{b}\alpha)=1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} &= \sum_{\substack{\alpha \in (q^{-1}\mathfrak{b}^{-1})^+ / E^+ \\ (q, q\mathfrak{b}\alpha)=1}} \sum_{i=0}^{\lambda-1} \frac{\chi(q\mathfrak{b}\alpha\epsilon^i)}{N(q\mathfrak{b}\alpha\epsilon^i)^s} \\ &= \lambda \cdot \sum_{\substack{\alpha \in (q^{-1}\mathfrak{b}^{-1})^+ / E^+ \\ (q, q\mathfrak{b}\alpha)=1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}. \end{aligned}$$

And from Lemma 2.2, we have

$$\begin{aligned} \sum_{\substack{\alpha \in (q^{-1}\mathfrak{b}^{-1})^+ / E_q^+ \\ (q, q\mathfrak{b}\alpha) = 1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} &= \sum_{(C,D) \in F} \sum_{\substack{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+ \\ (q, q\mathfrak{b}\alpha) = 1}} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} \\ &= \sum_{(C,D) \in F} \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}. \end{aligned}$$

By equation (2), the above is equal to

$$\sum_{(C,D) \in \tilde{F}'} \sum_{i=0}^{\lambda-1} \sum_{\alpha \in (\frac{(C+D\delta)\epsilon^i}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}.$$

Since

$$\sum_{\alpha \in (\frac{(C+D\delta)\epsilon^i}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s} = \sum_{\alpha \in (\frac{(C+D\delta)}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{\chi(q\mathfrak{b}\alpha\epsilon^i)}{N(q\mathfrak{b}\alpha\epsilon^i)^s},$$

the above also equal to

$$\lambda \cdot \sum_{(C,D) \in \tilde{F}'} \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{\chi(q\mathfrak{b}\alpha)}{N(q\mathfrak{b}\alpha)^s}.$$

Note that for $\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+$, $q\mathfrak{b}\alpha$ and $(C+D\delta)\mathfrak{b}$ are in the same ray class modulo q . Thus $\chi(q\mathfrak{b}\alpha) = \chi((C+D\delta)\mathfrak{b})$. This completes the proof. \square

2.1. Shintani-Zagier cone decomposition. We review briefly the decomposition of $(\mathbb{R}^2)^+$ into cones due to Shintani and Zagier in [18], [19], [20]. This depends on a real quadratic field K and a fixed ideal \mathfrak{a} inside. Here for the sake of computation, we fix $\mathfrak{a} = \mathfrak{b}^{-1}$ where \mathfrak{b} is set as in the beginning of this section.

K is embedded into \mathbb{R}^2 by $\iota = (\tau_1, \tau_2)$, where τ_1, τ_2 are two real embeddings of K . Especially the totally positive elements of K lands on $(\mathbb{R}^2)^+$. We are going to describe the fundamental domain of $(\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+$ embedded into $(\mathbb{R}^2)^+$.

The multiplicative action of E_q^+ on K^+ induces an action on $(\mathbb{R}^2)^+$ by coordinate-wise multiplication:

$$\epsilon \circ (x, y) = (\tau_1(\epsilon)x, \tau_2(\epsilon)y).$$

A fundamental domain $\mathfrak{D}_{\mathbb{R}}$ of $(\mathbb{R}^2)^+ / E_q^+$ is given by

$$(3) \quad \mathfrak{D}_{\mathbb{R}} := \{x\iota(1) + y\iota(\epsilon^{-\lambda}) \mid x > 0, y \geq 0\} \subset (\mathbb{R}^2)^+$$

where $E_q^+ = \langle \epsilon^\lambda \rangle$ for an integer λ and $\epsilon > 1$ is the unique totally positive fundamental unit.

If we take the convex hull of $\iota(\mathfrak{b}^{-1}) \cap (\mathbb{R}^2)^+$ in $(\mathbb{R}^2)^+$, the vertices on the boundary are $\{P_i\}_{i \in \mathbb{Z}}$ for $P_i \in \iota(\mathfrak{b}^{-1})$ and determined by the inequalities that $P_0 = \iota(1)$, $P_{-1} = \iota(\delta)$ and $x(P_i) < x(P_{i-1})$ where $x(P_k)$ denotes the first coordinate of P_k for $k \in \mathbb{Z}$. Since any two consecutive boundary points make a basis of $\iota(\mathfrak{b}^{-1})$, we find that

$$\begin{pmatrix} 0 & 1 \\ -1 & b_i \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \end{pmatrix} = \begin{pmatrix} P_i \\ P_{i+1} \end{pmatrix},$$

for an integer b_i . It is easy to see that $b_i \geq 2$ from the convexity. Thus we obtain

$$(4) \quad x(P_{i-1}) + x(P_{i+1}) = b_i x(P_i).$$

Put $\delta_i := \frac{x(P_{i-1})}{x(P_i)} > 1$. Note that $\delta_0 = \delta$. δ_i satisfies a recursive relation:

$$\delta_i = b_i - \frac{1}{\delta_{i+1}}, \quad \text{for } i \in \mathbb{Z}.$$

Therefore

$$\delta_i = b_i - \frac{1}{b_{i+1} - \frac{1}{b_{i+2} - \dots}} = (b_i, b_{i+1}, b_{i+2}, \dots).$$

Let $\epsilon > 1$ be the totally positive fundamental unit. ϵ moves a boundary point to another boundary point preserving the order. Thus we have

$$(5) \quad \epsilon \circ P_i = P_{i-m},$$

for a positive integer m . Therefore we obtain the following proposition.

Proposition 2.4. (1) $\delta_{i+m} = \delta_i$ for all $i \in \mathbb{Z}$.

$$(2) \quad \delta_i = ((b_i, b_{i+1}, \dots, b_{i+m-1})) = b_i - \frac{1}{b_{i+1} - \dots - \frac{1}{b_{i+m-1} - \frac{1}{b_i - \dots}}}.$$

$$(3) \quad \iota(\epsilon^{-1}) = P_m$$

$$(4) \quad \epsilon^{-1} \circ P_i = P_{i+m}$$

$$(5) \quad \iota(\epsilon^{-\gamma}) = P_{\gamma m}$$

Proof. (1) $\delta_{i+m} = \frac{x(P_{i+m-1})}{x(P_{i+m})} = \frac{\epsilon x(P_{i-1})}{\epsilon x(P_i)} = \delta_i$.

(2) This is an immediate consequence of 1.

(3) From Eq. (5),

$$P_m = \epsilon^{-1} \circ P_0.$$

Since $P_0 = \iota(1)$ and $\epsilon^{-1} \circ \iota(1) = \iota(\epsilon^{-1})$.

(4) This is immediate from (5).

(5) It is trivial from (3) and (4). \square

From (3) and (4) of Proposition 2.4, $\mathfrak{D}_{\mathbb{R}}$ the fundamental domain $(\mathbb{R}^2)^+/E_q^+$ is further decomposed into $(\lambda \cdot m)$ -disjoint union of smaller cones:

$$\mathfrak{D}_{\mathbb{R}} = \bigsqcup_{i=1}^{\lambda m} \{xP_{i-1} + yP_i \mid x > 0, y \geq 0\}.$$

Obviously the fundamental set of the quotient $(\iota(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}) \cap (\mathbb{R}^2)^+)/E_q^+$ inside $\mathfrak{D}_{\mathbb{R}}$, which we denote by \mathfrak{D} is given by a disjoint union:

$$\mathfrak{D} := \bigsqcup_{i=1}^{\lambda m} \left(\iota\left(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\right) \cap \{xP_{i-1} + yP_i \mid x > 0, y \geq 0\} \right).$$

Since $\{P_{i-1}, P_i\}$ is a \mathbb{Z} -basis of $\iota(\mathfrak{b}^{-1})$, there is a unique $(x_{C+D\delta}^i, y_{C+D\delta}^i) \in (0, 1] \times [0, 1)$ such that

$$x_{C+D\delta}^i P_{i-1} + y_{C+D\delta}^i P_i \in \iota\left(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\right),$$

for each $i, C, D \in \mathbb{Z}$. Thus

$$\begin{aligned} (6) \quad & \iota\left(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\right) \cap \{xP_{i-1} + yP_i \mid x > 0, y \geq 0\} \\ &= \{(x_{C+D\delta}^i + n_1)P_{i-1} + (y_{C+D\delta}^i + n_2)P_i \mid n_1, n_2 \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

In [16], Yamamoto found a recursive relation satisfied by $(x_{C+D\delta}^i, y_{C+D\delta}^i)$:

$$\begin{aligned} (7) \quad & x_{C+D\delta}^{i+1} = \langle b_i x_{C+D\delta}^i + y_{C+D\delta}^i \rangle, \\ & y_{C+D\delta}^{i+1} = 1 - x_{C+D\delta}^i, \end{aligned}$$

where $\langle \cdot \rangle$ is as defined at the end of the introduction. (ie. $\langle x \rangle = x - [x]$ (resp. 1) for $x \notin \mathbb{Z}$ (resp. for $x \in \mathbb{Z}$)). ((2.1.3) of *loc. sit.*).

Let $A_i := x(P_i)$ for all $i \in \mathbb{Z}$. Then from Eq.(6), we obtain the following:

$$\begin{aligned} (8) \quad & \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+/E_q^+} \frac{1}{N(\alpha)^s} \\ &= \sum_{i=1}^{\lambda m} \sum_{n_1, n_2 \geq 0} N((x_{C+D\delta}^i + n_1)A_{i-1} + (y_{C+D\delta}^i + n_2)A_i)^{-s} \\ &= \sum_{i=1}^{\lambda m} \sum_{n_1, n_2 \geq 0} N((x_{C+D\delta}^i + n_1)\delta_i + (y_{C+D\delta}^i + n_2))^{-s} A_i^{-s}. \end{aligned}$$

In [19], Shintani evaluated $\sum_{n_1, n_2 \geq 0} N((x + n_1)\delta + (y + n_2))^{-s}$ at nonpositive integers. In particular, the value at $s = 0$ is expressed by first and second Bernoulli polynomials as follows:

Lemma 2.5 (Shintani).

$$\begin{aligned} & \sum_{n_1, n_2 \geq 0} N((x + n_1)\delta + (y + n_2))^{-s} \Big|_{s=0} \\ &= \frac{\delta + \delta'}{4} B_2(x) + B_1(x)B_1(y) + \frac{1}{4} \left(\frac{1}{\delta} + \frac{1}{\delta'} \right) B_2(y). \end{aligned}$$

Using this, we have

$$\begin{aligned} (9) \quad & \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{1}{N(\alpha)^s} \Big|_{s=0} \\ &= \sum_{i=1}^{\lambda m} \frac{\delta_i + \delta'_i}{4} B_2(x_{C+D\delta}^i) + B_1(x_{C+D\delta}^i) B_1(y_{C+D\delta}^i) + \frac{1}{4} \left(\frac{1}{\delta_i} + \frac{1}{\delta'_i} \right) B_2(y_{C+D\delta}^i) \end{aligned}$$

Moreover, Yamamoto in the proof of Theorem 4.1.1 of [16] simplified the above:

Lemma 2.6 (Yamamoto).

$$\begin{aligned} & \sum_{i=1}^{\lambda m} \frac{\delta_i + \delta'_i}{4} B_2(x_{C+D\delta}^i) + \frac{1}{4} \left(\frac{1}{\delta_i} + \frac{1}{\delta'_i} \right) B_2(y_{C+D\delta}^i) \\ &= \sum_{i=1}^{\lambda m} \frac{b_i}{2} B_2(x_{C+D\delta}^i) \end{aligned}$$

Finally, we have

$$\begin{aligned} (10) \quad & \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1})^+ / E_q^+} \frac{1}{N(\alpha)^s} \Big|_{s=0} \\ &= \sum_{i=1}^{\lambda m} B_1(x_{C+D\delta}^i) B_1(y_{C+D\delta}^i) + \frac{b_i}{2} B_2(x_{C+D\delta}^i) \end{aligned}$$

Lemma 2.7. *Let ϵ be the totally positive fundamental unit of K and $\lambda := [E^+ : E_q^+]$. Then we have*

$$x_{C+D\delta}^{mi+j} = x_{\epsilon^i(C+D\delta)}^j \quad \text{and} \quad y_{C+D\delta}^{mi+j} = y_{\epsilon^i(C+D\delta)}^j,$$

for $j = 0, 1, 2, \dots, m-1$.

Proof. From (4) of Proposition 2.4, we have

$$A_{mi+j} = \epsilon^{-i} A_j,$$

for any integer i .

Thus

$$\begin{aligned} x_{C+D\delta}^{mi+j} A_{mi+j-1} + y_{C+D\delta}^{mi+j} A_{mi+j} &= \\ x_{C+D\delta}^{mi+j} \epsilon^{-i} A_{j-1} + y_{C+D\delta}^{mi+j} \epsilon^{-i} A_j &\in \frac{C+D\delta}{q} + \mathfrak{b}^{-1}. \end{aligned}$$

Therefore,

$$x_{C+D\delta}^{mi+j} A_{j-1} + y_{C+D\delta}^{mi+j} A_j \in \frac{\epsilon^i \cdot (C+D\delta)}{q} + \mathfrak{b}^{-1}.$$

□

From Lemma 2.7 and the periodicity of b_i , we have

Lemma 2.8.

$$\begin{aligned} &\sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1}) + / E_q^+} \frac{1}{N(\alpha)^s} \Big|_{s=0} \\ &= \sum_{i=1}^m \sum_{j=0}^{\lambda-1} B_1(x_{\epsilon^j * (C+D\delta)}^i) B_1(y_{\epsilon^j * (C+D\delta)}^i) + \frac{b_i}{2} B_2(x_{\epsilon^j * (C+D\delta)}^i). \end{aligned}$$

Finally, we have

Proposition 2.9. *For a ray class character χ modulo q and an ideal \mathfrak{b} of K such that*

$$\mathfrak{b}^{-1} = [1, \delta]$$

for $\delta \in K$ with $\delta > 2$ and $0 < \delta' < 1$, we have

$$\begin{aligned} &L_K(0, \chi, \mathfrak{b}) \\ &= \sum_{1 \leq C, D \leq q} \chi((C+D\delta)\mathfrak{b}) \sum_{i=1}^m B_1(x_{(C+D\delta)}^i) B_1(y_{(C+D\delta)}^i) + \frac{b_i}{2} B_2(x_{(C+D\delta)}^i) \end{aligned}$$

Proof. From Proposition 2.3, we obtain

$$\begin{aligned} &L_K(0, \chi, \mathfrak{b}) \\ &= \sum_{(C,D) \in \tilde{F}'} \chi((C+D\delta)\mathfrak{b}) \sum_{\alpha \in (\frac{C+D\delta}{q} + \mathfrak{b}^{-1}) + / E_q^+} N(q\mathfrak{b}\alpha)^{-s} \Big|_{s=0}. \end{aligned}$$

Lemma 2.8 implies that the above is equal to

$$\sum_{(C,D) \in \tilde{F}'} \chi((C+D\delta)\mathfrak{b}) \sum_{j=0}^{\lambda-1} \sum_{i=1}^m B_1(x_{\epsilon^j*(C+D\delta)}^i) B_1(y_{\epsilon^j*(C+D\delta)}^i) + \frac{b_i}{2} B_2(x_{\epsilon^j*(C+D\delta)}^i).$$

Since $(C+D\delta)\epsilon\mathfrak{b} = (C+D\delta)\mathfrak{b}$, the above is expressed as follows

$$\sum_{(C,D) \in \tilde{F}'} \sum_{j=0}^{\lambda-1} \chi((C+D\delta)\epsilon^j\mathfrak{b}) \sum_{i=1}^m B_1(x_{\epsilon^j*(C+D\delta)}^i) B_1(y_{\epsilon^j*(C+D\delta)}^i) + \frac{b_i}{2} B_2(x_{\epsilon^j*(C+D\delta)}^i).$$

From

$$(11) \quad F = \bigsqcup_{i=0}^{\lambda-1} \epsilon^i \tilde{F}',$$

we find that the above equals to

$$\sum_{(C,D) \in F} \chi((C+D\delta)\mathfrak{b}) \sum_{i=1}^m B_1(x_{(C+D\delta)}^i) B_1(y_{(C+D\delta)}^i) + \frac{b_i}{2} B_2(x_{(C+D\delta)}^i).$$

If $((C+D\delta)\mathfrak{b}, q) \neq 1$ then $\chi((C+D\delta)\mathfrak{b}) = 0$. Thus we complete the proof. \square

Remark 2.10. It is important to note that the summation running over $C, D \in [1, q]$ is actually supported on F . This is justified by the twist of the mod q Dirichlet character. Obviously, F depends on δ in K , but the twisted sum has invariant form of δ and K . This is a subtle point in the proof of the main theorem as we deal with family of the Hecke's L -values with respect to a family $(K_n, \chi_n, \mathfrak{b})$.

3. PROOF OF THE MAIN THEOREM

In this section, we compute the special values of Hecke's L-function for a family of real quadratic fields. The computation is made using the expression of the L-value in the previous section. After the computation, it will be apparent that the linearity property comes sufficiently from the shape of the continued fractions in the family. This will complete the proof of Theorem 1.3.

This gives a criterion that will recover several approaches of class number problems for some families of real quadratic fields.

Consider a family of real quadratic fields $K_n = \mathbb{Q}(\sqrt{d_n})$, where d_n is a positive square free integer. For a fixed Dirichlet character χ of modulus q , we associate a ray class character $\chi_n := \chi \circ N_{K_n/\mathbb{Q}}$ for each

n . Let us fix an ideal \mathfrak{b}_n of K_n for each n . Then we have a family of the Hecke's L-functions associated to $(K_n, \chi_n, \mathfrak{b}_n)$:

$$L_{K_n}(s, \chi_n, \mathfrak{b}_n) = \sum_{\mathfrak{a}} \frac{\chi_n(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where \mathfrak{a} runs over integral ideals \mathfrak{a} in the ray class represented by \mathfrak{b}_n .

3.1. Plan of the proof. Assume that

$$\mathfrak{b}_n^{-1} = [1, \delta(n)]$$

with $\delta(n) > 2, 0 < \delta(n)' < 1$. As discussed in Prop.2.4, $\delta(n)$ has a purely periodic minus continued fraction expansion:

$$\begin{aligned} \delta(n) &= ((b_0(n), b_1(n), \dots, b_{m(n)-1}(n))) \\ (12) \quad &= b_0(n) - \frac{1}{b_1(n) - \dots - \frac{1}{b_{m(n)-1}(n) - \frac{1}{b_0(n) - \dots}}} \end{aligned}$$

with $b_k(n) \geq 2$.

We extend the definition of $b_i(n)$ for all $i \in \mathbb{Z}$ by requiring that $b_{i+m(n)}(n) = b_i(n)$ for $i \in \mathbb{Z}$. Let $\delta_k(n) = ((b_k(n), b_{k+1}(n), \dots, b_{k+m(n)-1}(n)))$ and we define $\{A_k(n)\}_{k \in \mathbb{Z}}$ by

$$A_{-1}(n) = \delta(n), A_0(n) = 1, \dots, A_{k+1}(n) = A_k(n) / \delta_{k+1}(n).$$

Then for fixed C, D and n , there is a unique $(x_{C+D\delta(n)}^i, y_{C+D\delta(n)}^i)$ such that

$$(13) \quad 0 < x_{C+D\delta(n)}^i \leq 1, \quad 0 \leq y_{C+D\delta(n)}^i < 1,$$

$$(14) \quad x_{C+D\delta(n)}^i A_{i-1}(n) + y_{C+D\delta(n)}^i A_i(n) \in \frac{C + D\delta(n)}{q} + \mathfrak{b}_n^{-1},$$

for each $i \in \mathbb{Z}$, as described in the previous section. This $(x_{C+D\delta(n)}^i, y_{C+D\delta(n)}^i)$ satisfies Yamamoto's recursive relation (7) as follows:

$$(15) \quad x_{C+D\delta(n)}^{i+1} = \langle b_i(n) x_{C+D\delta(n)}^i + y_{C+D\delta(n)}^i \rangle, \quad y_{C+D\delta(n)}^{i+1} = 1 - x_{C+D\delta(n)}^i.$$

Now we recall a standard conversion formula of a plus continued fraction expansion to minus continued fraction expansion:

Lemma 3.1. *Let $\delta - 1$ be a purely periodic continued fraction:*

$$[[a_0, a_1, \dots, a_{s-1}]].$$

Then the minus continued fraction expansion of δ is

$$((b_0, b_1, \dots, b_{m-1})),$$

where

$$b_i := \begin{cases} a_{2j} + 2, & \text{for } i = S_j \\ 2, & \text{otherwise} \end{cases}$$

for

$$S_j = \begin{cases} 0, & \text{for } j = 0 \\ S_{j-1} + a_{2j-1}, & \text{for } j \geq 1 \end{cases}$$

and the period

$$m = \begin{cases} a_1 + a_3 + a_5 \cdots + a_{s-1} = S_{\frac{s}{2}}, & \text{for even } s \\ a_0 + a_1 + a_2 \cdots + a_{s-1} = S_s, & \text{for odd } s \end{cases}$$

Proof. (See page 177, 178 of [18]). Actually if s is an odd integer, the period m is

$$\sum_{i=1}^s a_{2i-1} = a_1 + a_3 + \cdots + a_{2s-1} = S_s.$$

Since a_i has period s , we find that

$$a_1 + 1_3 + \cdots + a_{2s-1} = a_0 + a_1 + a_2 \cdots + a_{s-1} = \sum_{i=0}^{s-1} a_i.$$

□

For the family of $\delta(n) \in K$, we assumed that

$$\delta(n) - 1 = [[a_0(n), a_1(n), a_2(n), \dots, a_{s-1}(n)]],$$

has the same period for every n .

Then $\delta(n)$ has purely periodic minus continued fraction expansion

$$\delta(n) = ((b_0(n), b_1(n), \dots, b_{m(n)-1}(n)))$$

with $b_i(n)$, $S_j(n)$ and $m(n)$ defined by the same manner as in the previous lemma.

One should be aware that $m(n)$ vary with n , while the period of positive continued fraction s is fixed.

From Proposition 2.9 and recursive relation (15) of $(x_{C+D\delta(n)}^i, y_{C+D\delta(n)}^i)$, we have

(16)

$$L_{K_n}(0, \chi_n, \mathbf{b}_n) =$$

$$\sum_{1 \leq C, D \leq q} \chi_n((C + D\delta(n))\mathbf{b}_n) \sum_{i=1}^{m(n)} \left(B_1(x_{C+D\delta(n)}^i) B_1(y_{C+D\delta(n)}^i) + \frac{b_i(n)}{2} B_2(x_{C+D\delta(n)}^i) \right).$$

To check the linear behavior, it suffices to show that

$$(17) \quad \sum_{i=1}^{m(n)} (B_1(x_{C+D\delta(n)}^i)B_1(y_{C+D\delta(n)}^i) + \frac{b_i(n)}{2}B_2(x_{C+D\delta(n)}^i))$$

is linear in k with the coefficients determined only by r .

Because $b_i(n) = 2$ if $i \neq S_j(n)$ for some j , we can divide the above into two parts:

$$(18) \quad \begin{aligned} & \sum_{l=1}^{s\mu(s)} \left(-B_1(x_{C+D\delta(n)}^{S_l(n)})B_1(x_{C+D\delta(n)}^{S_l(n)-1}) + \frac{a_{2l}(n)+2}{2}B_2(x_{C+D\delta(n)}^{S_l(n)}) \right) \\ & + \sum_{l=0}^{s\mu(s)-1} \sum_{i=S_l(n)+1}^{S_{l+1}(n)-1} F(x_{C+D\delta(n)}^i, x_{C+D\delta(n)}^{i-1}) \end{aligned}$$

where $\mu(s) = \frac{1}{2}$ or 1 for s even or odd, respectively, and $F(x, y) := -B_1(x)B_1(y) + B_2(x)$.

If C, D are fixed and there is no danger of misunderstanding, $x_i(n)$ will simply mean $x_{C+D\delta(n)}^i$.

Below is the behavior of $x_i(n)$, when n varies. The proof will be given later.

1. $\{x_i(n)\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ is an arithmetic progression mod \mathbb{Z} with common difference $\langle x_{S_j(n)+1}(n) - x_{S_j(n)}(n) \rangle$.
2. $\{x_i(n)\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ has period q .
3. $x_{S_j(n)}(n)$, $x_{S_j(n)-1}(n)$ and $x_{S_j(n)+1}(n)$ are invariant as k varies for $n = qk + r$.

In short, $\{x_i(n)\}$ is a ‘piecewise arithmetic progression’.

As we have constrained that $a_i(n) = \alpha_i n + \beta_i$, $\langle a_i(n) \rangle_q$ is independent of k for $n = qk + r$ but depends only on i and r .

Define $\gamma_i(r)$ as follows:

$$(19) \quad \gamma_i(r) := \langle a_i(n) \rangle_q$$

Then actually $\gamma_i(r)$ is $\langle a_i(r) \rangle_q$. Since $\{F(x_i(n), x_{i-1}(n))\}_{S_j(n)+1 \leq i \leq S_{j+1}(n)-1}$ has period q from 2 above, we obtain

$$\begin{aligned}
 (20) \quad & \sum_{i=S_l(n)+1}^{S_{l+1}(n)-1} F(x_i(n), x_{i-1}(n)) \\
 &= \sum_{i=S_l(n)+1}^{S_l(n)+\gamma_{2l+1}(r)-1} F(x_i(n), x_{i-1}(n)) + \kappa_{2l+1}(n) \sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_i(n), x_{i-1}(n))
 \end{aligned}$$

where $a_i(n) = \kappa_i(n)q + \gamma_i(r)$ for an integer $\kappa_i(n)$. Written precisely,

$$(21) \quad \kappa_i(n) = \frac{a_i(n) - \gamma_i(r)}{q}.$$

Since

$$\alpha_i r + \beta_i = q\tau_i(r) + \gamma_i(r)$$

for some integer $\tau_i(r)$, we can write for $n = qk + r$

$$(22) \quad \kappa_i(n) = k\alpha_i + \tau_i(r)$$

Using 3, $x_{S_l(n)}(n)$ and $x_{S_l(n)+1}(n)$ are sufficiently determined by the residue r of n by q . *A priori* the summations $\sum_{i=S_l(n)+1}^{S_l(n)+\gamma_{2l+1}(r)-1} F(x_i(n), x_{i-1}(n))$ and $\sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_i(n), x_{i-1}(n))$ are completely determined by $x_{S_l(n)}(n)$ and $x_{S_l(n)+1}(n)$ and remain unchanged while k varies.

Thus we conclude first that

I. For $n = qk + r$, $\sum_{i=S_l(n)+1}^{S_{l+1}(n)-1} F(x_i(n), x_{i-1}(n))$ is linear function of k .

Using (21) and (22), we have

$$\begin{aligned}
 (23) \quad & -B_1(x_{S_l(n)}(n))B_1(x_{S_l(n)-1}(n)) + \frac{a_{2l}(n) + 2}{2} B_2(x_{S_l(n)}(n)) \\
 &= -B_1(x_{S_l(n)}(n))B_1(x_{S_l(n)-1}(n)) + \frac{\alpha_{2l}qk + \tau_{2l}(r)q + \gamma_{2l}(r) + 2}{2} B_2(x_{S_l(n)}(n))
 \end{aligned}$$

Again after 3 we conclude that

II. For $n = qk + r$, $-B_1(x_{S_l(n)}(n))B_1(x_{S_l(n)-1}(n)) + \frac{a_{2l}(n)+2}{2} B_2(x_{S_l(n)}(n))$ is linear function on k .

Additionally, we have

III. s and $\mu(s)$ is independent of n .

Altogether I, II and III imply the linearity of $\sum_{i=1}^{m(n)} -B_1(x_i(n))B_1(x_{i-1}(n)) + \frac{b_i(n)}{2}B_2(x_i(n))$ in k and the coefficients are function in r for fixed C, D .

In sequel, we will clarify the properties 1, 2, 3 of $\{x_i(n)\}$. Also we give precise description $\sum_{i=1}^{m(n)} -B_1(x_i(n))B_1(x_{i-1}(n)) + \frac{b_i(n)}{2}B_2(x_i(n))$ that will finish the proof of Theorem 1.3.

3.2. Periodicity and invariance. In this section, we will prove the properties 1, 2, 3 of $\{x_i(n)\}$ in the previous section.

Proposition 3.2. *For $j \geq 0$, $\{x_i(n)\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ is an arithmetic progression mod \mathbb{Z} with common difference $\langle x_{S_j(n)+1}(n) - x_{S_j(n)}(n) \rangle$.*

Proof. Since $b_i(n) = 2$ for $S_j(n) + 1 \leq i \leq S_{j+1}(n) - 1$, we have that

$$x_{i+1}(n) = \langle 2x_i(n) - x_{i-1}(n) \rangle.$$

It implies that for $S_j(n) + 1 \leq i \leq S_{j+1}(n) - 1$,

$$\langle x_{i+1}(n) - x_i(n) \rangle = \langle \langle 2x_i(n) - x_{i-1}(n) \rangle - x_i(n) \rangle = \langle x_i(n) - x_{i-1}(n) \rangle.$$

□

Lemma 3.3. *For $i \geq -1$, we have*

- (1) $qx_i(n) \in \mathbb{Z}$.
- (2) $0 < x_i(n) \leq 1$.

Proof. Since $A_0(n) = 1$, $A_{-1}(n) = \delta(n)$, from (13), (14) and (15) we find that

$$x_0(n) = \left\langle \frac{D}{q} \right\rangle, \quad x_{-1}(n) = 1 - \frac{C}{q}.$$

We also note that $b_i(n) \in \mathbb{Z}$ for any $i \geq 0$. Thus (15) implies above lemma. □

Proposition 3.4. *For $j \geq 0$ and $a_{2j+1}(n) \geq q$, $\{x_i(n)\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ has period q . Explicitly we have*

$$x_{S_j(n)+q+i}(n) = x_{S_j(n)+i}(n) \text{ for } 0 \leq i \leq a_{2j+1}(n) - q.$$

Proof. Note that $\{x_i(n) \bmod 1\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ is an arithmetic progression. Thus we have

$$x_{S_j(n)+q+i}(n) = \langle x_{S_j(n)+i}(n) + q \langle x_{S_j(n)+i}(n) - x_{S_j(n)+i-1}(n) \rangle \rangle,$$

for $0 \leq i \leq a_{2j+1}(n) - q$. From Lemma 3.3, we find that

$$q \langle x_{S_j(n)+i}(n) - x_{S_j(n)+i-1}(n) \rangle \in \mathbb{Z}.$$

Thus

$$\langle x_{S_j(n)+i}(n) + q \langle x_{S_j(n)+i}(n) - x_{S_j(n)+i-1}(n) \rangle \rangle = \langle x_{S_j(n)+i}(n) \rangle.$$

Since $0 < x_{S_j(n)+i}(n) \leq 1$, we finally have that

$$\langle x_{S_j(n)+i}(n) \rangle = x_{S_j(n)+i}(n).$$

□

For $0 \leq r \leq q-1$, we define

$$\Gamma_j(r) := \begin{cases} 0, & \text{for } j = 0 \\ \Gamma_j(r) + \gamma_{2j-1}(r), & \text{for } j \geq 1 \end{cases},$$

where $\gamma_i(r)$ is defined as in (19). For $i \geq 0$, we put

$$c_i(r) = \begin{cases} \gamma_{2j}(r) + 2, & \text{for } i = \Gamma_j(r) \\ 2, & \text{otherwise} \end{cases}$$

Consider a sequence $\{\nu_{CD}^i(r)\}_{i \geq -1}$ with the initial value and the recursive relation as follows:

$$\nu_{CD}^{-1}(r) = \frac{q-C}{q}, \quad \nu_{CD}^0(r) = \langle \frac{D}{q} \rangle$$

and

$$\nu_{CD}^{i+1}(r) = \langle c_i(r) \nu_{CD}^i(r) - \nu_{CD}^{i-1}(r) \rangle.$$

If C, D are fixed and clear from the context, we omit the subscript and abbreviate $\nu_{CD}^i(r)$ to $\nu_i(r)$.

Proposition 3.5. *With the notations above, for $j \geq 0$ and $n = qk + r$, we have*

$$x_{S_j(n)+i}(n) = \nu_{\Gamma_j(r)+i}(r) \quad \text{for } 0 \leq i \leq \gamma_{2j+1}(r)$$

Proof. We use induction on j .

When $j = 0$. $S_0(n) = \Gamma_0(r) = 0$. We need to show $x_i(n) = \nu_i(r)$ for $i \in [0, \gamma_1(r)]$. As we have seen in the proof of lemma 3.3,

$$x_0(n) = \langle \frac{D}{q} \rangle = \nu_0(r), \quad x_{-1}(n) = 1 - \frac{C}{q} = \nu_{-1}(r).$$

Since $a_0(n) - \gamma_0(r) \in q\mathbb{Z}$, using (15) and the recursive relation of $\nu_i(r)$, one can easily check that

$$x_1(n) = \langle (a_0(n) + 2) \langle \frac{D}{q} \rangle + \frac{C}{q} \rangle = \langle (\gamma_0(r) + 2) \nu_0(r) - \nu_{-1}(r) \rangle = \nu_1(r)$$

For $1 \leq i \leq \gamma_1(r) - 1$, $x_i(n)$ and $\nu_i(r)$ satisfy the same recursive relation

$$x_{i+1}(n) = \langle 2x_i(n) - x_{i-1}(n) \rangle, \quad \nu_{i+1}(r) = \langle 2\nu_i(r) - \nu_{i-1}(r) \rangle.$$

Thus we have

$$x_i(n) = \nu_i(r) \quad \text{for } 0 \leq i \leq \gamma_1(r).$$

Now assume that the proposition holds true for $j < j_0$. From Proposition 3.4, we find that if $a_{2j_0-1}(n) \geq q$ then

$$(24) \quad x_{S_{j_0-1}(n)+q+i}(n) = x_{S_{j_0-1}(n)+i}(n) \text{ for } 0 \leq i \leq a_{2j_0-1}(n) - q.$$

Since $a_{2j_0-1}(n) - \gamma_{2j_0-1}(r) \in q\mathbb{Z}$, we obtain

$$\begin{aligned} x_{S_{j_0}(n)-1}(n) &= x_{S_{j_0-1}(n)+a_{2j_0-1}(n)-1}(n) = x_{S_{j_0-1}(n)+\gamma_{2j_0-1}(r)-1}(n) \\ &= \nu_{\Gamma_{j_0-1}(r)+\gamma_{2j_0-1}(r)-1}(r) = \nu_{\Gamma_{j_0}(r)-1}(r). \end{aligned}$$

and

$$\begin{aligned} x_{S_{j_0}(n)}(n) &= x_{S_{j_0-1}(n)+a_{2j_0-1}(n)}(n) \\ &= x_{S_{j_0-1}(n)+\gamma_{2j_0-1}(r)}(n) = \nu_{\Gamma_{j_0-1}(r)+\gamma_{2j_0-1}(r)}(r) = \nu_{\Gamma_{j_0}(r)}(r). \end{aligned}$$

Moreover from (15), we find that

$$(25) \quad \begin{aligned} x_{S_{j_0}(n)+1}(n) &= \langle (a_{2j_0}(n) + 2)x_{S_{j_0}(n)}(n) - x_{S_{j_0}(n)-1}(n) \rangle \\ &= \langle (\gamma_{2j_0}(r) + 2)\nu_{\Gamma_{j_0}(r)}(r) - \nu_{\Gamma_{j_0}(r)-1}(r) \rangle = \nu_{\Gamma_{j_0}(r)+1}(r). \end{aligned}$$

Since for $S_{j_0}(n) + 1 \leq i \leq S_{j_0+1}(n) - 1$

$$x_{i+1}(n) = \langle 2x_i(n) - x_{i-1}(n) \rangle$$

and for $\Gamma_{j_0}(r) + 1 \leq i \leq \Gamma_{j_0}(r) + \gamma_{2j_0+1}(r) - 1 = \Gamma_{j_0+1}(r) - 1$,

$$\nu_{i+1}(r) = \langle 2\nu_i(r) - \nu_{i-1}(r) \rangle,$$

we have

$$x_{S_{j_0}(n)+i}(n) = \nu_{\Gamma_{j_0}(r)+i}(r) \text{ for } 0 \leq i \leq \gamma_{2j_0+1}(r).$$

□

3.3. Summations. In this section we express $\sum_{i=1}^{m(n)} -B_1(x_{C+D\delta(n)}^i)B_1(x_{C+D\delta(n)}^{i-1}) + \frac{b_i(n)}{2}B_2(x_{C+D\delta(n)}^i)$ using $\{\nu_{CD}^i(r)\}$.

Lemma 3.6. *Let $d_l(r) := \langle \nu_{\Gamma_l(r)+1}(r) - \nu_{\Gamma_l(r)}(r) \rangle$ and $[x]_1 := x - \langle x \rangle$. Then for $1 \leq \gamma \leq q$ and n such that $\gamma \leq a_{2l+1}(n)$ and $n = qk + r$, we have*

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} (x_i(n) - x_{i-1}(n))^2 = \gamma d_l(r)^2 + (1 - 2d_l(r))[\nu_{\Gamma_l(r)}(r) + d_l(r)\gamma]_1$$

Proof. Since $0 < x_i(n) \leq 1$, we have

$$-1 < x_i(n) - x_{i-1}(n) < 1.$$

Thus

$$x_i(n) - x_{i-1}(n) = \langle x_i(n) - x_{i-1}(n) \rangle + \psi_i(n),$$

where

$$\psi_i(n) = \begin{cases} -1, & x_i(n) \leq x_{i-1}(n) \\ 0, & x_i(n) > x_{i-1}(n). \end{cases}$$

As

$$\langle x_{i+1}(n) - x_i(n) \rangle = \langle \langle 2x_i(n) - x_{i-1}(n) \rangle - x_i(n) \rangle = \langle x_i(n) - x_{i-1}(n) \rangle$$

for $S_l(n) + 1 \leq i \leq S_{l+1}(n) - 1$, we have

$$\langle x_i(n) - x_{i-1}(n) \rangle = \langle x_{S_l(n)+1}(n) - x_{S_l(n)}(n) \rangle = \langle \nu_{\Gamma_l(r)+1}(r) - \nu_{\Gamma_l(r)}(r) \rangle = d_l(r).$$

Hence we have

$$x_i(n) - x_{i-1}(n) = d_l(r) + \psi_i(n).$$

Thus we obtain

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} (x_i(n) - x_{i-1}(n))^2 = \gamma d_l(r)^2 + (1 - 2d_l(r)) \sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \psi_i(n)^2.$$

Note that $\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \psi_i(n)^2$ equals the number of i 's satisfying $x_i(n) \leq x_{i-1}(n)$ for $S_l(n) + 1 \leq i \leq S_l(n) + \gamma$.

Therefore

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \psi_i(n)^2 = [x_{S_l(n)}(n) + d_l(r)\gamma]_1 = [\nu_{\Gamma_l(r)}(r) + d_l(r)\gamma]_1.$$

□

For simplicity, we let

$$F(x, y) := -B_1(x)B_1(y) + B_2(x) = (x - \frac{1}{2})(\frac{1}{2} - y) + x^2 - x + \frac{1}{6}.$$

Lemma 3.7. *If $l \geq 0$ and $a_{2l+1}(n) \geq q$,*

$$\sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_i(n), x_{i-1}(n)) = \frac{1}{12} \left[6 \left(q d_l(r)^2 + (1 - 2d_l(r)) [\nu_{\Gamma_l(r)}(r) + d_l(r)q]_1 \right) - q \right]$$

And if $1 \leq \gamma \leq q - 1$ and $a_{2l+1}(n) \geq \gamma$,

$$\begin{aligned} & \sum_{i=S_l(n)+1}^{S_l(n)+\gamma} F(x_i(n), x_{i-1}(n)) \\ &= \frac{1}{12} \left[6 \left(\gamma d_l(r)^2 + (1 - 2d_l(r)) [\nu_{\Gamma_l(r)}(r) + d_l(r)\gamma]_1 + B_2(x_{S_l(n)+\gamma}(n)) - B_2(x_{S_l(n)}(n)) \right) - \gamma \right] \end{aligned}$$

where $B_2(x)$ is the second Bernoulli polynomial.

Proof. We note that

$$F(x, y) = \frac{1}{2}(x - y)^2 - \frac{1}{12} + \frac{1}{2}(B_2(x) - B_2(y)).$$

Thus we have

$$\begin{aligned} & \sum_{i=S_l(n)+1}^{S_l(n)+\gamma} F(x_i(n), x_{i-1}(n)) \\ &= \sum_{i=S_l(n)+1}^{S_l(n)+\gamma} \left[\frac{1}{2}(x_i(n) - x_{i-1}(n))^2 - \frac{1}{12} + \frac{1}{2}(B_2(x_i(n)) - B_2(x_{i-1}(n))) \right]. \end{aligned}$$

We note that for $1 \leq \gamma \leq q - 1$,

$$\sum_{i=S_l(n)+1}^{S_l(n)+\gamma} B_2(x_i(n)) - B_2(x_{i-1}(n)) = B_2(x_{S_l(n)+\gamma}(n)) - B_2(x_{S_l(n)}(n)).$$

and for $\gamma = q$ from the periodicity of $x_i(n)$ we have

$$\sum_{i=S_l(n)+1}^{S_l(n)+q} B_2(x_i(n)) - B_2(x_{i-1}(n)) = 0.$$

□

Proposition 3.8. *Suppose $\delta(n) - 1 = [[a_0(n), a_2(n), \dots, a_{s-1}(n)]]$, $a_i(n) = \alpha_i n + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{Z}$ and $a_i(r) = q\tau_i(r) + \gamma_i(r)$ for $\gamma_i(r) = \langle a_i(r) \rangle_q$. Let $d_{CD}^l(r) := \langle \nu_{CD}^{\Gamma_l(r)+1}(r) - \nu_{CD}^{\Gamma_l(r)}(r) \rangle$. Then, for $n = qk + r$, we have*

$$\sum_{i=1}^{m(n)} -B_1(x_{C+D\delta(n)}^i) B_1(y_{C+D\delta(n)}^i) + \frac{b_i(n)}{2} B_2(x_{C+D\delta(n)}^i) = \frac{1}{12} (A_{CD}(r) + k B_{CD}(r))$$

where

$$\begin{aligned} A_{CD}(r) &:= \sum_{l=1}^{s\mu(s)} -12B_1(\nu_{CD}^{\Gamma_l(r)}(r)) B_1(\nu_{CD}^{\Gamma_l(r)-1}(r)) + 6(a_{2l}(r) + 2) B_2(\nu_{CD}^{\Gamma_l(r)}(r)) \\ &+ \sum_{l=0}^{s\mu(s)-1} \left[6 \left((\gamma_{2l+1}(r) - 1) d_{CD}^l(r)^2 + (1 - 2d_{CD}^l(r)) [\nu_{CD}^{\Gamma_l(r)}(r) + d_{CD}^l(r) (\gamma_{2l+1}(r) - 1)]_1 \right. \right. \\ &\left. \left. + B_2(\nu_{CD}^{\Gamma_{l+1}(r)-1}(r)) - B_2(\nu_{CD}^{\Gamma_l(r)}(r)) \right) - \gamma_{2l+1}(r) + 1 \right. \\ &\left. + \tau_{2l+1}(r) \left(6(qd_{CD}^l(r)^2 + (1 - 2d_{CD}^l(r)) [\nu_{CD}^{\Gamma_l(r)}(r) + d_{CD}^l(r) q]_1) - q \right) \right] \end{aligned}$$

and

(26)

$$B_{CD}(r) := \sum_{l=1}^{s\mu(s)} 6q\alpha_{2l}B_2(\nu_{CD}^{\Gamma_l(r)}(r)) \\ + \sum_{l=0}^{s\mu(s)-1} \alpha_{2l+1} \left(6(qd_{CD}^l(r))^2 + (1 - 2d_{CD}^l(r))[\nu_{CD}^{\Gamma_l(r)} + d_{CD}^l(r)q]_1 - q \right)$$

Proof. From equation (18), we have

$$\sum_{i=1}^{m(n)} B_1(x_{C+D\delta(n)}^i)B_1(y_{C+D\delta(n)}^i) + \frac{b_i(n)}{2}B_2(x_{C+D\delta(n)}^i) \\ = \sum_{l=1}^{s\mu(s)} [-B_1(x_{C+D\delta(n)}^{S_l(n)})B_1(x_{C+D\delta(n)}^{S_l(n)-1}) + \frac{\alpha_{2l}qk + \tau_{2l}(r)q + \gamma_{2l}(r) + 2}{2}B_2(x_{C+D\delta(n)}^{S_l(n)})] \\ + \sum_{l=0}^{s\mu(s)-1} \sum_{i=S_l(n)+1}^{S_l(n)+q\alpha_{2l+1}k + q\tau_{2l+1}(r) + \gamma_{2l+1}(r)-1} F(x_{C+D\delta(n)}^i, x_{C+D\delta(n)}^{i-1})$$

From lemma 3.7, we have

$$12 \sum_{i=S_l(n)+1}^{S_l(n)+q\alpha_{2l+1}k + q\tau_{2l+1}(r) + \gamma_{2l+1}(r)-1} F(x_{C+D\delta(n)}^i, x_{C+D\delta(n)}^{i-1}) = 12 \sum_{i=S_l(n)+1}^{S_l(n)+\gamma_{2l+1}(r)-1} F(x_{C+D\delta(n)}^i, x_{C+D\delta(n)}^{i-1}) \\ + 12(\alpha_{2l+1}k + \tau_{2l+1}(r)) \sum_{i=S_l(n)+1}^{S_l(n)+q} F(x_{C+D\delta(n)}^i, x_{C+D\delta(n)}^{i-1}) = \\ 6 \left((\gamma_{2l+1}(r) - 1)d_{CD}^l(r)^2 + (1 - 2d_{CD}^l(r))[\nu_{CD}^{\Gamma_l(r)}(r) + d_{CD}^l(r)(\gamma_{2l+1}(r) - 1)]_1 \right. \\ \left. + B_2(x_{C+D\delta(n)}^{S_l(n)+\gamma_{2l+1}-1}) - B_2(x_{C+D\delta(n)}^{S_l(n)}) \right) - (\gamma_{2l+1}(r) - 1) \\ + (\alpha_{2l+1}k + \tau_{2l+1}(r)) \left(6(qd_{CD}^l(r))^2 + (1 - 2d_{CD}^l(r))[\nu_{CD}^{\Gamma_l(r)}(r) + d_{CD}^l(r)q]_1 - q \right)$$

Since $x_{C+D\delta(n)}^{S_l(n)} = \nu_{CD}^{\Gamma_l(r)}(r)$, $x_{C+D\delta(n)}^{S_l(n)-1} = \nu_{CD}^{\Gamma_l(r)-1}(r)$ and $x_{C+D\delta(n)}^{S_l(n)+\gamma_{2l+1}(r)-1} = \nu_{CD}^{\Gamma_{l+1}(r)-1}$, we complete the proof. \square

3.4. End of the proof.

Proof. Since $\nu_{CD}^{\Gamma_l(r)}(r)$, $\nu_{CD}^{\Gamma_l(r)-1}(r)$ and $d_{CD}^l(r)$ are in $\frac{1}{q}\mathbb{Z}$, we find that

$$q^2 A_{CD}(r), q^2 B_{CD}(r) \in \mathbb{Z}.$$

Moreover, we have

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{12q^2} \sum_{C,D} \chi_n(C + D\delta(n))(q^2 A_{CD}(r) + kq^2 B_{CD}(r)).$$

Since χ is a Dirichlet character of modulus q , if $n = qk + r$, we can write

$$\chi_n(\mathfrak{b}_n(C + D\delta(n))) = F_{CD}(r),$$

for a function F_{CD} . Note if K_r is defined, $\chi_n(\mathfrak{b}_n(C + D\delta(n))) = \chi_r(\mathfrak{b}_r(C + D\delta(r))) = F_{CD}(r)$. We warn the reader that the above expression does not make sense if K_r and $\delta(r)$ are undefined.

If we set

$$A_\chi(r) := \sum_{C,D} F_{CD}(r) q^2 A_{CD}(r)$$

and

$$B_\chi(r) := \sum_{C,D} F_{CD}(r) q^2 B_{CD}(r),$$

we obtain the proof. □

4. BIRÓ'S METHOD

Let K_n be a family of real quadratic fields such that special value at $s = 0$ of the Hecke L -function has linearity. In [2] and [3], Biró developed a way to find the residue of n with $h(K_n) = 1$ by certain primes using the linearity. In this section, we sketch Biró's method.

Let $K_n = \mathbb{Q}(\sqrt{d})$ for a square free integer $d = f(n)$ and D_n be the discriminant K_n . For an odd Dirichlet character $\chi : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{C}^*$, χ_n denotes the ray class character defined as $\chi_n = \chi \circ N_{K_n} : I_n(q)/P_n(q)^+ \rightarrow \mathbb{C}^*$. $\chi_D = (\frac{D}{\cdot})$ denotes the Kronecker character. Then the special value of the Hecke L -function at $s = 0$ has a factorization

$$\begin{aligned} L_{K_n}(0, \chi_n) &= L(0, \chi) L(0, \chi \chi_{D_n}) \\ (27) \quad &= \left(\frac{1}{q} \sum_{a=1}^q a \chi(a) \right) \left(\frac{1}{q D_n} \sum_{b=1}^{q D_n} b \chi(b) \chi_{D_n}(b) \right), \end{aligned}$$

Let $\mathfrak{b}_n = O_{K_n}$. Suppose that $L_{K_n}(0, \chi_n, \mathfrak{b}_n)$ is linear in the form:

$$L_{K_n}(0, \chi_n, \mathfrak{b}_n) = \frac{1}{12q^2} (A_\chi(r) + k B_\chi(r)).$$

for $A_\chi(r), B_\chi(r) \in \mathbb{Z}[\chi(1), \chi(2) \cdots \chi(q)]$. Let ϵ_n be the fundamental unit of K_n . From Proposition 2.2 in [9], we find that $L_{K_n}(0, \chi_n, \mathfrak{b}_n) =$

$L_{K_n}(0, \chi_n, (\epsilon_n)\mathfrak{b}_n)$. Thus if the class number of K_n is one then we have for $n = qk + r$

$$L_{K_n}(0, \chi_n) = \frac{c}{12q^2}(A_\chi(r) + kB_\chi(r))$$

where c is the number of narrow ideal classes.

Then we have

$$B_\chi(r)k + A_\chi(r) = \frac{12q}{c} \cdot \left(\sum_{a=1}^q a\chi(a) \right) \cdot \left(\frac{1}{qD_n} \sum_{b=1}^{qD_n} b\chi(b)\chi_{D_n}(b) \right).$$

Let L_χ be the cyclotomic field generated by the values of χ . Since $\frac{1}{qD_n} \sum_{b=1}^{qD_n} b\chi(b)\chi_{D_n}(b)$ is integral in L_χ , for a prime ideal I of L_χ dividing $(\sum_{a=1}^q a\chi(a))$, we have

$$B_\chi(r)k + A_\chi(r) \equiv 0 \pmod{I}.$$

And if $I \nmid B_\chi(r)$ then

$$k \equiv -\frac{A_\chi(r)}{B_\chi(r)} \pmod{I}.$$

Since $n = qk + r$, we have

$$n \equiv -q\frac{A_\chi(r)}{B_\chi(r)} + r \pmod{I}.$$

Moreover if $O_{L_\chi}/I = \mathbb{Z}/p\mathbb{Z}$, the residue of n modulo p is expressed only in $A_\chi(r)$, $B_\chi(r)$ and r as above.

Below we arrange all the necessary conditions of q and p .

Condition(*)

1. q : odd integer
2. p : odd prime
3. χ : character with conductor q
4. I : prime ideal in L_χ lying over p
 $I | (\sum_{a=1}^q a\chi(a))$ and $O_{L_\chi}/I = \mathbb{Z}/p\mathbb{Z}$

Note that the condition is independent of the family $\{K_n\}$, once the linearity holds.

Let S be the set of (q, p) satisfying Condition(*).

$$S = \bigcup_{q:\text{odd integer}} S_q$$

where $S_q := \{(q, p) \in S\}$.

Finally we remark that for $(q, p) \in S$ we obtain the residue of $n = qk + r$ modulo p for which the class number of K_n is one.

The above method has been applied to find an upper bound of the discriminant of real quadratic fields with class number one in some families of Richaud-degert type where the linearity criterion is satisfied (cf. [3], [2], [7], [12]). Together with properly developed class number one criteria for each cases, the class number problems could be solved.

It is easily checked that in fact the criterion is fulfilled by general families of Richaud-Degert type. Furthermore, there are still abundant examples such families of real quadratic fields satisfying the linearity criterion (cf. [14]). For these, we have controlled behavior of the special values of Hecke's L -function at $s = 0$ and Biro's method is directly applied for each cases. We expect many other meaningful problems for family of real quadratic fields than class number problem in arithmetic can be studied in this line.

5. A GENERALIZATION

We conclude this section with a possible generalization of the linearity of the special value of the Hecke L -function. This generalization will be dealt in a separate paper [10].

As in the criterion for linearity, we set $K_n = \mathbb{Q}(\sqrt{f(n)})$ and \mathfrak{b}_n is an integral ideal of K_n . We assume

$$\mathfrak{b}_n^{-1} = [1, \delta(n)],$$

for

$$\delta(n) - 1 = [a_1(n), a_2(n), \dots, a_s(n)]$$

with $a_i(x) \in \mathbb{Z}[x]$.

For a given conductor q , write $n = qk + r$ for $r = 0, 1, 2, \dots, q - 1$. Suppose $N = \max_i \{\deg(a_i(x))\}$, then we obtain that the special value of the partial ζ -function of the ray class of $\mathfrak{b}_n \bmod q$ at $s = 0$ is written as

$$\zeta_{K_{n,q}}(0, (C + D\delta(n))\mathfrak{b}_n) = \frac{1}{12q^2} (A_0(r) + A_1(r)k + \dots + A_N(r)k^N)$$

for some rational integers A_i depending only on r .

We have no application of this property in arithmetic. It will be very interesting if one applies this in similar fashion as Biró's method as presented here.

REFERENCES

- [1] Barnes, E.W., *On the theory of the multiple gamma function*, Tran. Cambridge Phil. Soc., **19** (1904), 374-425.
- [2] Biró, A., *Yokoi's conjecture*, Acta Arith. **106.1** (2003), 85-104.
- [3] Biró, A., *Chowla's conjecture*, Acta Arith. **107.2** (2003), 179-194.

- [4] Byeon, D. and Kim, H., *Class number 2 criteria for real quadratic fields of Richaud-Degert type*, J. Number Theory **62**, 257-272 (1997).
- [5] Byeon, D. and Kim, H., *Class number 1 criteria for real quadratic fields of Richaud-Degert type*, J. Number Theory **57**, 328-339 (1996).
- [6] Chowla, S. and Friedlander, J., *Class numbers and quadratic residues*, Glasgow Math. J. **17** (1976), 47-52.
- [7] Byeon, D., Kim, M. and Lee, J., *Mollin's conjecture*, Acta Arithmetica (2007)
- [8] Byeon, D. and Lee, J., *Class number 2 problem for certain real quadratic fields of Richaud-Degert type*, J. Number Theory **128** 865-883
- [9] Byeon, D. and Lee, J., *A complete determination of Rabinowitch polynomials*, preprint 20 pages
- [10] Jun, B. and Lee, J., *Special values of partial zeta functions of real quadratic fields at $s = 0$* , in preparation.
- [11] Lee, J., *The complete determination of narrow Richaud-Degert type which is not 5 modulo 8 with class number two*, J. Number Theory **129** 604-620.
- [12] Lee, J., *The complete determination of wide Richaud-Degert type which is not 5 modulo 8 with class number one*, to appear in Acta Arithmetica.
- [13] Mollin, R.A., *Quadratics*, CRC Press (1996)
- [14] McLaughlin, J., *Polynomial solutions of Pell's equation and Fundamental units in Real quadratic fields*, J. London Math Soc **67**, 2003, 16-28.
- [15] Mollin, R.A., *Class number one criteria for real quadratic fields I*, Proc. Japan Acad **63**, 1987, 121-125.
- [16] Yamamoto, S., *On Kronecker limit formulas for real quadratic fields*, J. Number Theory. **128**, 2008, no.2, 426-450.
- [17] Yokoi, H., *Class number one problem for certain kind of real quadratic fields*, in: Proc. Internat. Conf. (Katata, 1986), Nagoya Univ., Nagoya, 1986, 125-137.
- [18] Zagier, D., *A Kronecker Limit Formula for Real Quadratic Fields*, Math. Ann. **213**, 153-184 (1975).
- [19] Shintani, T., *On special values of zeta functions of totally real algebraic number fields at non-positive integers*, J. Fac.Sci.Univ. Tokyo. **63** (1976), 393-417.
- [20] Van der Geer, G., *Hilbert Modular Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **16** Springer-Verlag (1980).

E-mail address: byungheup@gmail.com

E-mail address: lee9311@kias.re.kr

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, HOE-GIRO 87, DONGDAEMUN-GU, SEOUL 130-722, KOREA